Introduction to Mathematics and Modeling

lecture 8

The cross product
This week

1. Section 12.4: the cross product
2. Section 12.5: lines and planes in space
The cross product – introduction

Definition

Let \( \mathbf{u} = (u_1, u_2, u_3) \) and \( \mathbf{v} = (v_1, v_2, v_3) \) be two vectors in \( \mathbb{R}^3 \). The cross product \( \mathbf{u} \times \mathbf{v} \) is defined as

\[
\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).
\]
The cross product – introduction

Section 12.4

1.1

Definition

Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be two vectors in $\mathbb{R}^3$. The cross product van $u$ and $v$ is defined as

$$u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

- The Dutch name for the cross product is *uitproduct* or *uitwendig product*. 
Definition

Let \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) be two vectors in \( \mathbb{R}^3 \). The cross product \( u \times v \) is defined as

\[
\begin{align*}
    u \times v &= (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) \end{align*}
\]

- The Dutch name for the cross product is \textit{uitproduct} or \textit{uitwendig product}.
- The cross product can be computed using this trick:

\[
\begin{pmatrix}
    u_1 & u_2 & u_3 \\
    v_1 & v_2 & v_3
\end{pmatrix}
\]
Definition

Let \( \mathbf{u} = (u_1, u_2, u_3) \) and \( \mathbf{v} = (v_1, v_2, v_3) \) be two vectors in \( \mathbb{R}^3 \). The cross product van \( \mathbf{u} \) and \( \mathbf{v} \) is defined as

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\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).
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  u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3
\end{pmatrix}
\begin{pmatrix}
u_1 \\
v_2
\end{pmatrix}
\]
The cross product – introduction

Definition

Let \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) be two vectors in \( \mathbb{R}^3 \). The cross product van \( u \) and \( v \) is defined as

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\begin{pmatrix}
\mathbf{u} & \mathbf{v}
\end{pmatrix} = \begin{pmatrix}
    u_1 & u_2 & u_3 \\
    v_1 & v_2 & v_3
\end{pmatrix}
\begin{pmatrix}
    u_1 & u_2 \\
    v_1 & v_2
\end{pmatrix}
\]
The cross product – introduction

Definition

Let \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) be two vectors in \( \mathbb{R}^3 \). The cross product van \( u \) and \( v \) is defined as

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- The Dutch name for the cross product is **uitproduct** or **uitwendig product**.
- The cross product can be computed using this trick:

\[
\mathbf{u} \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} \mathbf{v}
\]

Exercise

Calculate the cross product \( \mathbf{u} \times \mathbf{v} \) of \( \mathbf{u} = (2, 2, -1) \) and \( \mathbf{v} = (-1, 2, 2) \).

Answer

\[
\mathbf{u} \times \mathbf{v} = (6, -3, 6).
\]
Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be two vectors in $\mathbb{R}^3$. The cross product van $\mathbf{u}$ and $\mathbf{v}$ is defined as

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Exercise

Calculate the cross product $\mathbf{u} \times \mathbf{v}$ of $\mathbf{u} = (2, 2, -1)$ and $\mathbf{v} = (-1, 2, 2)$.

$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$.

- The Dutch name for the cross product is *uitproduct* or *uitwendig product*.
- The cross product can be computed using this trick:

$$
\mathbf{u} \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
**Exercise**

Calculate the cross product \( u \times v \) of \( u = (2, 2, -1) \) and \( v = (-1, 2, 2) \).

**Answer**

\[ u \times v = (6, -3, 6) \]

The cross product can be computed using this trick:

\[
\begin{bmatrix}
  u_1 & u_2 & u_3 \\
  v_1 & v_2 & v_3
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3
\end{bmatrix}
\begin{bmatrix}
  v_1 \\
  v_2
\end{bmatrix}
\]
Laws and properties

**Theorem**

*For all* \( u, v, w \in \mathbb{R}^n \) *and* \( r, s \in \mathbb{R} \) *we have*

1. \((ru) \times (sv) = (rs)(u \times v)\)
2. \(u \times (v + w) = u \times v + u \times w\)
3. \(u \times v = -(v \times u)\)
4. \((v + w) \times u = v \times u + w \times u\)
5. \(0 \times u = u \times 0 = 0\)
6. \(u \times (v \times w) = (u \cdot w)v - (u \cdot v)w\)

Property 4 can be proved with properties 2 and 3.
**Theorem**

For all $u, v, w \in \mathbb{R}^n$ and $r, s \in \mathbb{R}$ we have

1. $(ru) \times (sv) = (rs)(u \times v)$
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5. $0 \times u = u \times 0 = 0$
6. $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$

- Property 4 can be proved with properties 2 and 3.
Theorem

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors. If $\theta$ is the acute positive angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta.$$
Theorem

Let \( \mathbf{u} \) and \( \mathbf{v} \) be two vectors. If \( \theta \) is the acute positive angle between \( \mathbf{u} \) and \( \mathbf{v} \), then

\[
|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta.
\]

- Acute means: \( \theta \leq \pi \), hence \( \sin \theta \geq 0 \).
Theorem

For all vectors $u$ and $v$ we have $u \times v \perp u$ and $u \times v \perp v$. 

Vector $u \times v$ is perpendicular to the plane through $u$ and $v$. 

The length of $u \times v$ is $|u||v|\sin \theta$. 

The right-hand rule determines the direction of $u \times v$. 

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Theorem

For all vectors $\mathbf{u}$ and $\mathbf{v}$ we have $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}$ and $\mathbf{u} \times \mathbf{v} \perp \mathbf{v}$.

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**Theorem**

For all vectors \( \mathbf{u} \) and \( \mathbf{v} \) we have \( \mathbf{u} \times \mathbf{v} \perp \mathbf{u} \) and \( \mathbf{u} \times \mathbf{v} \perp \mathbf{v} \).

- Vector \( \mathbf{u} \times \mathbf{v} \) is perpendicular to the plane through \( \mathbf{u} \) and \( \mathbf{v} \).
- The length of \( \mathbf{u} \times \mathbf{v} \) is \( |\mathbf{u}| |\mathbf{v}| \sin \theta \).
The cross product – geometry

**Theorem**

*For all vectors \( \mathbf{u} \) and \( \mathbf{v} \) we have \( \mathbf{u} \times \mathbf{v} \perp \mathbf{u} \) and \( \mathbf{u} \times \mathbf{v} \perp \mathbf{v} \).*

- Vector \( \mathbf{u} \times \mathbf{v} \) is perpendicular to the plane through \( \mathbf{u} \) and \( \mathbf{v} \).
- The length of \( \mathbf{u} \times \mathbf{v} \) is \( |\mathbf{u}| |\mathbf{v}| \sin \theta \).
- The right-hand rule determines the direction of \( \mathbf{u} \times \mathbf{v} \).
The area of a parallelogram

**Theorem**

Let \( u \in \mathbb{R}^3 \) and \( v \in \mathbb{R}^3 \) be the edges of a parallelogram \( P \). Then the area of \( P \) is equal to \( |u \times v| \).
The area of a parallelogram

**Theorem**

Let $u \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ be the edges of a parallelogram $P$. Then the area of $P$ is equal to $|u \times v|$.

- Observe that $\sin \theta = \frac{h}{|v|}$, so $h = |v| \sin \theta$. 

The area of a parallelogram

**Theorem**

Let \( u \in \mathbb{R}^3 \) and \( v \in \mathbb{R}^3 \) be the edges of a parallelogram \( P \). Then the area of \( P \) is equal to \( |u \times v| \).

- Observe that \( \sin \theta = \frac{h}{|v|} \), so \( h = |v| \sin \theta \).
- The area of \( P \) is

\[
|u| \ h = |u| \ |v| \sin \theta = |u \times v|.
\]
Example

Find the area of the triangle $D$ with vertices $P = (1, -1, 0)$, $Q = (2, 1, -1)$ and $R = (-1, 1, 2)$.
The area of a parallelogram

Example

Find the area of the triangle $D$ with vertices $P = (1, -1, 0)$, $Q = (2, 1, -1)$ and $R = (-1, 1, 2)$.

- The triangle is one half of a parallelogram with edges $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, hence the area of $D$ is

$$\frac{1}{2} \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right|.$$
The area of a parallelogram

**Example**

*Find the area of the triangle $D$ with vertices $P = (1, -1, 0)$, $Q = (2, 1, -1)$ and $R = (-1, 1, 2)$.*

- The triangle is one half of a parallelogram with edges $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, hence the area of $D$ is

  \[
  \frac{1}{2} \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right |.
  \]

- For the cross product we have

  \[
  \overrightarrow{PQ} \times \overrightarrow{PR} = (1, 2, -1) \times (-2, 2, 2) = (6, 0, 6).
  \]
Find the area of the triangle $D$ with vertices $P = (1, -1, 0)$, $Q = (2, 1, -1)$ and $R = (-1, 1, 2)$.

- The triangle is one half of a parallelogram with edges $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, hence the area of $D$ is
  \[ \frac{1}{2} \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right|. \]

- For the cross product we have
  \[ \overrightarrow{PQ} \times \overrightarrow{PR} = (1, 2, -1) \times (-2, 2, 2) = (6, 0, 6). \]

- For the area we have
  \[ \text{area}(D) = \frac{1}{2} \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right| = \frac{1}{2} \sqrt{36 + 36} = 3\sqrt{2}. \]
The area of a parallelogram in $\mathbb{R}^2$

**Theorem**

Let $P$ be the parallelogram spanned by $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. Then $\text{area}(P) = |u_1v_2 - u_2v_1|$. 

![Diagram of parallelogram in 2D and 3D space]
The area of a parallelogram in $\mathbb{R}^2$

**Theorem**

Let $P$ be the parallelogram spanned by $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. Then $\text{area}(P) = |u_1 v_2 - u_2 v_1|$.

By appending a zero to the vectors $\mathbf{u}$ and $\mathbf{v}$ we can embed $P$ in $\mathbb{R}^3$:

$\mathbf{u}' = (u_1, u_2, 0)$ and $\mathbf{v}' = (v_1, v_2, 0)$
The area of a parallelogram in \( \mathbb{R}^2 \)

**Theorem**

Let \( P \) be the parallelogram spanned by \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \). Then \( \text{area}(P) = |u_1 v_2 - u_2 v_1| \).

By appending a zero to the vectors \( u \) and \( v \) we can embed \( P \) in \( \mathbb{R}^3 \):

\[
u' = (u_1, u_2, 0) \quad \text{and} \quad v' = (v_1, v_2, 0)
\]

The area of \( P \) is

\[
\text{area } P = |u' \times v'| = |(0, 0, u_1 v_2 - u_2 v_1)| = |u_1 v_2 - u_2 v_1|.
\]
Problem

Let $S$ be a point in space and let $\ell$ be a line through $P$ with direction vector $v$. Find the distance $d$ of $S$ to $\ell$. 

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Method 1: Use the projection of $u = \overrightarrow{PS}$ on $\ell$:

Works in $\mathbb{R}^n$ for every $n$

$$d = |h| = \left| u - \frac{u \cdot v}{v \cdot v} v \right|$$
Problem

Let $S$ be a point in space and let $\ell$ be a line through $P$ with direction vector $v$. Find the distance $d$ of $S$ to $\ell$.

Method 1: Use the projection of $u = \overrightarrow{PS}$ on $\ell$:

$$d = |h| = \left| u - \frac{u \cdot v}{v \cdot v} v \right|$$

Works in $\mathbb{R}^n$ for every $n$

Method 2: Use the cross product:

$$d = |u| \sin \theta = \frac{|u \times v|}{|v|}$$

Only works in $\mathbb{R}^3$

Section 12.5, formula (5)
Example

Find the distance of \( S = (1, 1, 5) \) to the line

\[ \ell : \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t. \]

Using method 2:
Example

Find the distance of \( S = (1, 1, 5) \) to the line

\[
\ell : \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.
\]

Using method 2:

- Define \( P = (1, 3, 0), \quad \overrightarrow{OP} = (1, 3, 0) \) and \( \mathbf{v} = (1, -1, 2) \), then

\[
\ell : \quad \mathbf{p} + t\mathbf{v} \quad (t \in \mathbb{R}).
\]
Distance to a line

Example

Find the distance of \( S = (1, 1, 5) \) to the line

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\ell : \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.
\]

Using method 2:

- Define \( P = (1, 3, 0) \), \( p = \overrightarrow{OP} = (1, 3, 0) \) and \( v = (1, -1, 2) \), then \( \ell : p + tv \ (t \in \mathbb{R}) \).
- Define \( u = \overrightarrow{PS} = (1, 1, 5) - (1, 3, 0) = (0, -2, 5) \).
Example

Find the distance of \( S = (1, 1, 5) \) to the line

\[
\ell : \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.
\]

Using method 2:

- Define \( P = (1, 3, 0) \), \( \mathbf{p} = \overrightarrow{OP} = (1, 3, 0) \) and \( \mathbf{v} = (1, -1, 2) \), then
  \[
  \ell : \quad \mathbf{p} + t\mathbf{v} \quad (t \in \mathbb{R}).
  \]
- Define \( \mathbf{u} = \overrightarrow{PS} = (1, 1, 5) - (1, 3, 0) = (0, -2, 5). \)
- \( \mathbf{v} \cdot \mathbf{v} = 1^2 + (-1)^2 + 2^2 = 6 \), hence \( |\mathbf{v}| = \sqrt{6}. \)
Example 5

Find the distance of $S = (1, 1, 5)$ to the line

\[ \ell : \ x = 1 + t, \quad y = 3 - t, \quad z = 2t. \]

Using method 2:

- Define $P = (1, 3, 0)$, $\mathbf{p} = \overrightarrow{OP} = (1, 3, 0)$ and $\mathbf{v} = (1, -1, 2)$, then $\ell : \mathbf{p} + t\mathbf{v} \ (t \in \mathbb{R})$.
- Define $\mathbf{u} = \overrightarrow{PS} = (1, 1, 5) - (1, 3, 0) = (0, -2, 5)$.
- $\mathbf{v} \cdot \mathbf{v} = 1^2 + (-1)^2 + 2^2 = 6$, hence $|\mathbf{v}| = \sqrt{6}$.
- $\mathbf{u} \times \mathbf{v} = (0, -2, 5) \times (1, -1, 2) = (1, 5, 2)$. 

The distance is $d = \frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}$. 

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Example

Find the distance of $S = (1, 1, 5)$ to the line

$$\ell : \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$  

Using method 2:

- Define $P = (1, 3, 0)$, $p = \overrightarrow{OP} = (1, 3, 0)$ and $v = (1, -1, 2)$, then $\ell : p + tv \ (t \in \mathbb{R})$.
- Define $u = \overrightarrow{PS} = (1, 1, 5) - (1, 3, 0) = (0, -2, 5)$.
- $v \cdot v = 1^2 + (-1)^2 + 2^2 = 6$, hence $|v| = \sqrt{6}$.
- $u \times v = (0, -2, 5) \times (1, -1, 2) = (1, 5, 2)$.
- The distance is

$$d = \frac{|u \times v|}{|v|}$$
Example

Find the distance of \( S = (1, 1, 5) \) to the line

\[
\ell : \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.
\]

Using method 2:

- Define \( P = (1, 3, 0) \), \( \mathbf{p} = \overrightarrow{OP} = (1, 3, 0) \) and \( \mathbf{v} = (1, -1, 2) \), then \( \ell : \mathbf{p} + t\mathbf{v} \ (t \in \mathbb{R}) \).
- Define \( \mathbf{u} = \overrightarrow{PS} = (1, 1, 5) - (1, 3, 0) = (0, -2, 5) \).
- \( \mathbf{v} \cdot \mathbf{v} = 1^2 + (-1)^2 + 2^2 = 6 \), hence \( |\mathbf{v}| = \sqrt{6} \).
- \( \mathbf{u} \times \mathbf{v} = (0, -2, 5) \times (1, -1, 2) = (1, 5, 2) \).
- The distance is

\[
d = \frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1^2 + 5^2 + 2^2}}{\sqrt{6}}
\]
Distance to a line

**Example**

*Find the distance of* $S = (1, 1, 5)$ *to the line*

\[ \ell : \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t. \]

**Using method 2:**

- Define $P = (1, 3, 0)$, $\mathbf{p} = \overrightarrow{OP} = (1, 3, 0)$ and $\mathbf{v} = (1, -1, 2)$, then
  \[ \ell : \mathbf{p} + t\mathbf{v} \ (t \in \mathbb{R}). \]
- Define $\mathbf{u} = \overrightarrow{PS} = (1, 1, 5) - (1, 3, 0) = (0, -2, 5)$.
- $\mathbf{v} \cdot \mathbf{v} = 1^2 + (-1)^2 + 2^2 = 6$, hence $|\mathbf{v}| = \sqrt{6}$.
- $\mathbf{u} \times \mathbf{v} = (0, -2, 5) \times (1, -1, 2) = (1, 5, 2)$.
- The distance is
  \[ d = \frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1^2 + 5^2 + 2^2}}{\sqrt{6}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}. \]
Assignment: IMM2 - Tutorial 8.1
**Definition**

A parametrisation of the plane $M$ is a function of the form

$$\mathbf{p} + s\mathbf{v} + t\mathbf{w}, \quad s, t \in \mathbb{R}$$

- The vector $\mathbf{p}$ is called a **support vector** and the vectors $\mathbf{v}$ and $\mathbf{w}$ are called **direction vectors**.
Example

Find a parametrisation of the plane through the points \( A = (0, 0, 1), \) \( B = (2, 0, 0) \) and \( C = (0, 3, 0) \).

\[
\text{Choose support vector } \mathbf{a} = \mathbf{OA} = (0, 0, 1).
\]

\[
\text{Choose direction vectors } \mathbf{v} = \mathbf{AB} = (2, 0, -1) \text{ and } \mathbf{w} = \mathbf{AC} = (0, 3, -1).
\]

A parametrisation then is

\[
r(s, t) = \mathbf{a} + s \mathbf{v} + t \mathbf{w} = (0, 0, 1) + s(2, 0, -1) + t(0, 3, -1) = (2s, 3t, 1 - s - t),
\]

\( s, t \in \mathbb{R} \).

Check: \( A = r(0, 0), B = r(1, 0) \) and \( C = r(0, 1) \).
Example

Find a parametrisation of the plane through the points $A = (0, 0, 1)$, $B = (2, 0, 0)$ and $C = (0, 3, 0)$.

Choose support vector $a = \overrightarrow{OA} = (0, 0, 1)$. 

Parametrisation of a plane

Example

Find a parametrisation of the plane through the points \( A = (0, 0, 1) \), \( B = (2, 0, 0) \) and \( C = (0, 3, 0) \).

- Choose support vector \( \mathbf{a} = \overrightarrow{OA} = (0, 0, 1) \).
- Choose direction vectors
  \[
  \mathbf{v} = \overrightarrow{AB} = (2, 0, -1)
  \]
  and
  \[
  \mathbf{w} = \overrightarrow{AC} = (0, 3, -1).
  \]
Example 7

**Example**

Find a parametrisation of the plane through the points $A = (0, 0, 1)$, $B = (2, 0, 0)$ and $C = (0, 3, 0)$.

- Choose support vector $\mathbf{a} = \overrightarrow{OA} = (0, 0, 1)$.
- Choose direction vectors
  
  $\mathbf{v} = \overrightarrow{AB} = (2, 0, -1)$
  
  and
  
  $\mathbf{w} = \overrightarrow{AC} = (0, 3, -1)$.

- A parametrisation then is
  
  $\mathbf{r}(s, t) = \mathbf{a} + s\mathbf{v} + t\mathbf{w}$
  
  $= (0, 0, 1) + s(2, 0, -1) + t(0, 3, -1)$
  
  $= (2s, 3t, 1 - s - t), \quad s, t \in \mathbb{R}$. 
Example 7

Find a parametrisation of the plane through the points $A = (0, 0, 1)$, $B = (2, 0, 0)$ and $C = (0, 3, 0)$.

- Choose support vector $a = \overrightarrow{OA} = (0, 0, 1)$.
- Choose direction vectors
  \[ v = \overrightarrow{AB} = (2, 0, -1) \]
  \[ w = \overrightarrow{AC} = (0, 3, -1). \]
- A parametrisation then is
  \[ r(s, t) = a + sv + tw \]
  \[ = (0, 0, 1) + s(2, 0, -1) + t(0, 3, -1) \]
  \[ = (2s, 3t, 1 - s - t), \quad s, t \in \mathbb{R}. \]
- Check: $A = r(0, 0)$, $B = r(1, 0)$ en $C = r(0, 1)$. 
Problem

Find an equation of a plane $M$ given by a parametrisation

$$p + sv + tw,$$

where $P$ is a point of $M$ and $p = \overrightarrow{OP}$.

Method 1: Three-point method: observe that $P$, $Q = p + v$ and $R = p + w$ are three points of $M$. This gives three equations involving $x$, $y$, $z$, $s$ and $t$. Eliminate $s$ and $t$ to find one equation in $x$, $y$ and $z$.

Method 2: Compute a normal vector $n = v \times w$ of $M$, then

$$M: n \cdot (x - p) = 0.$$
Example

Find an equation of the plane through the points \( A = (0, 0, 1) \), \( B = (2, 0, 0) \) and \( C = (0, 3, 0) \).
Example

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- A parametrisation of \( M \) is
  \[
  \mathbf{p} + s\mathbf{v} + t\mathbf{w} = (0, 0, 1) + s(2, 0, -1) + t(0, 3, -1)
  \]
Equation of a plane

Example

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- A parametrisation of $M$ is
  $$ p + sv + tw = (0, 0, 1) + s(2, 0, -1) + t(0, 3, -1) $$
- Find a normal vector:
  $$ n = v \times w = (2, 0, -1), 2, 0 $$
  $$ n = (0, 3, -1), 0, 3 $$
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- Find a normal vector:
  \[
  \mathbf{n} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix}
    2 & 0 & -1 \\
    0 & 3 & -1 \\
  \end{vmatrix} = (3, 2, 6).
  \]
- The normal equation of \( M \) is
  \[
  \mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0
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  \]
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  \[
  \mathbf{n} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix}
    \mathbf{i} & \mathbf{j} & \mathbf{k} \\
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    0 & 3 & -1
  \end{vmatrix} = (3, 2, 6).
  \]
- The normal equation of \( M \) is
  \[
  \mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0
  \]
  \[
  (3, 2, 6) \cdot \left( (x, y, z) - (0, 0, 1) \right) = 0
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  \[ 3x + 2y + 6(z - 1) = 0 \]
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  (0, 3, -1), 0, 3
  \]

- The normal equation of $M$ is
  \[
  n \cdot (x - p) = 0 \\
  (3, 2, 6) \cdot \left( (x, y, z) - (0, 0, 1) \right) = 0 \\
  3x + 2y + 6(z - 1) = 0 \\
  3x + 2y + 6z = 6
  \]
Theorem

Two different non-parallel planes intersect in a line.
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Theorem

Two different non-parallel planes intersect in a line.

- Non-parallel means: the normals of both planes have different directions.
- If the planes are called $M$ and $N$, then the intersection line is denoted as follows:
  \[ \ell = M \cap N. \]
- A line in space can be regarded as the intersection line of two planes, in other words: it is the solution of a system of two equations:
  \[ \ell: \begin{align*}
  ax + by + cz &= d, \\
  px + qy + rz &= s.
  \end{align*} \]
Example 8+9

Find a parametrisation of the intersection line of the planes

\[ 3x - 6y - 2z = 15 \quad \text{and} \quad 2x + y - 2z = 5. \]

**Method 1:**
Example

Find a parametrisation of the intersection line of the planes $\begin{align*}
3x - 6y - 2z &= 15 \\
2x + y - 2z &= 5.
\end{align*}$

Method 1:

- From the first equation follows $x = 2y + \frac{2}{3}z + 5.$
Example

Find a parametrisation of the intersection line of the planes

\[3x - 6y - 2z = 15 \text{ and } 2x + y - 2z = 5.\]

Method 1:

- From the first equation follows \( x = 2y + \frac{2}{3}z + 5.\)
- Substitution in the second equation gives
  \[
  2 \left(2y + \frac{2}{3}z + 5\right) + y - 2z = 5,
  \]
  and after simplification we have
  \[
  z = \frac{15}{2} y + \frac{15}{2}.
  \]
Example

**Example 8+9**

Find a parametrisation of the intersection line of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

**Method 1:**

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  and after simplification we have
  
  $$z = \frac{15}{2}y + \frac{15}{2}.$$
  
- Choose one of the unknowns as parameter. For example, let $y = t$, then
  
  $$z = \frac{15}{2}t + \frac{15}{2} \quad \text{and} \quad x = 2t + \frac{2}{3}\left(\frac{15}{2}t + \frac{15}{2}\right) + 5 = 7t + 10.$$
Intersection line of two planes

Example

Find a parametrisation of the intersection line of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Method 1:

- From the first equation follows $x = 2y + \frac{2}{3}z + 5$.
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- Choose one of the unknowns as parameter. For example, let $y = t$, then
  \[ z = \frac{15}{2} t + \frac{15}{2} \quad \text{and} \quad x = 2t + \frac{2}{3} \left( \frac{15}{2} t + \frac{15}{2} \right) + 5 = 7t + 10. \]
- A parametrisation of the intersection line is
  \[ \mathbf{r}(t) = \left(7t + 10, t, \frac{15}{2} t + \frac{15}{2}\right) = \left(10, 0, \frac{15}{2}\right) + t \left(7, 1, \frac{15}{2}\right), \quad t \in \mathbb{R}. \]
Method 2:

- The normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are perpendicular to the intersection line, so the cross product of $\mathbf{n}_1$ and $\mathbf{n}_2$ is a direction vector of the intersection line.
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- Extract the normal vectors from the equations:

  \[ M_1: 3x - 6y - 2z = 15, \quad \rightarrow \quad \mathbf{n}_1 = (3, -6, -2), \]

  \[ M_2: 2x + y - 2z = 5, \quad \rightarrow \quad \mathbf{n}_2 = (2, 1, -2), \]

  hence $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = (14, 2, 15)$. 
Intersection line of two planes

\[ M_1: 3x - 6y - 2z = 15, \quad \rightarrow \quad n_1 = (3, -6, -2), \]
\[ M_2: 2x + y - 2z = 5, \quad \rightarrow \quad n_2 = (2, 1, -2), \]
\[ v = (14, 2, 15) \]

A support vector can be found by choosing a value for \( x, y \) or \( z \), and then solving both equations for \( x \) and \( y \). For example, choose \( y = 0 \):
\[ 3x - 2z = 15, \]
\[ 2x - 2z = 5. \]
Intersection line of two planes

\[ M_1: 3x - 6y - 2z = 15, \quad \rightarrow n_1 = (3, -6, -2), \]
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Subtracting both equations gives \( x = 10 \), and therefore \( z = \frac{15}{2} \).
Intersection line of two planes

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- Subtracting both equations gives \(x = 10\), and therefore \(z = \frac{15}{2}\).
- A support vector is \(p = \left( 10, 0, \frac{15}{2} \right)\).
Intersection line of two planes

\[ M_1 : 3x - 6y - 2z = 15, \quad \rightarrow \quad \mathbf{n}_1 = (3, -6, -2), \]
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\[
3x - 2z = 15, \\
2x - 2z = 5.
\]
- Subtracting both equations gives \( x = 10 \), and therefore \( z = \frac{15}{2} \).
- A support vector is \( \mathbf{p} = \left( 10, 0, \frac{15}{2} \right) \).
- A parametrisation of the intersection line is
\[
\mathbf{p} + t \mathbf{v} = \left( 10, 0, \frac{15}{2} \right) + t \left( 14, 2, 15 \right)
= \left( 10, 0, \frac{15}{2} \right) + 2t \left( 7, 1, \frac{15}{2} \right).
\]
Assignment: IMM2 - Tutorial 8.2